Discrete Beamforming Optimization for RISs with a Limited Phase Range and Amplitude Attenuation

Dogan Kutay Pekcan, Hongyi Liao, and Ender Ayanoglu

***Abstract*—This paper addresses the problem of maximizing the received power at a user equipment via reconfigurable intelligent surface (RIS) characterized by phase-dependent amplitude (PDA) and discrete phase shifts over a limited phase range. Given complex RIS coefficients, that is, discrete phase shifts and PDAs, we derive the necessary and sufficient conditions to achieve the optimal solution. To this end, we propose an optimal search algorithm that is proven to converge in linear time within at most NK steps, significantly outperforming the exhaustive search approach that would otherwise be needed for RISs with amplitude attenuation. The proposed optimal algorithm provides a generic upper bound that could serve as a benchmark for discrete beamforming in RISs with amplitude constraints.**

***Index Terms*—Intelligent reflective surface (IRS), reconfigurable intelligent surface (RIS), nonuniform discrete phase shifts, IRS/RIS phase range, global optimum, linear time discrete beamforming for IRS/RIS**

I. Problem Definition

Consider a Reconfigurable Intelligent Surface (RIS) structure as in Fig. 1. This structure is intended to be used as a reflector for transmissions from a base station (BS) to a user equipment (UE) when the direct link from the BS to the UE may be blocked. Let the combination of the BS-to-RIS and the RIS-to-UE channels for the n-th RIS element be $h_n,\ n=1,2,\ldots,N$ where N is the number of individual RIS elements. We assume $h_n=\beta_n e^{j\alpha_n}$ with $\beta_n \geq 0$ and $\alpha \in [-\pi,\pi)$, and the n-th RIS element introduces a complex number $\beta^r(\theta_n)e^{j\theta_n}$ to h_n with $\beta^r(\theta_n)\in[0,1]$ and θ_n taking values from a discrete phase shift set $\Phi_K=\{\phi_1,\phi_2,\ldots,\phi_K\}$ where K is an integer, $n=1,2,\ldots,N$. We let $h_0=\beta_0 e^{j\alpha_0}$ be the direct link between the BS and UE. We state that $\beta^r(\theta_n):[-\pi,\pi]\to[0,1]$ is a function of θ_n representing the gain of the n-th RIS element. We refer to this dependence between the RIS phase shift θ_n and the respective gain $\beta^r(\theta_n)$ for the n-th RIS element as the phase dependent amplitude (PDA) model. To accommodate for the discrete phase shift constraint and the PDA model, we define $\mathbf{W}_K$, where $\mathbf{W}_K=\{\beta^r(\phi_1)e^{j\phi_1},\beta^r(\phi_2)e^{j\phi_2},\ldots,\beta^r(\phi_K)e^{j\phi_K}\}$, using the phases in Φ_K together with the respective gains. Therefore, RIS coefficients will belong to this coefficients set, i.e., $\mathrm{w}_n\in\mathbf{W}_K$ for $n=1,\ldots,N$. We also define the difference among each adjacent phase shift in Φ_K as $\Omega_K=\{\omega_1,\omega_2,\ldots,\omega_K\}$, such that $\phi_{k\oplus 1}=\phi_k+\omega_k$.[1]

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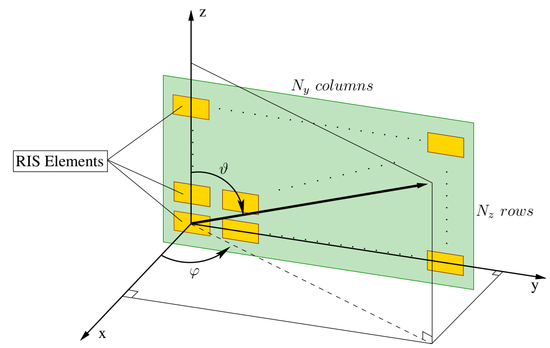

Fig. 1: RIS structure.

Our goal is to maximize the received power at the UE, that is, $|h_0+\sum_{n=1}^{N}h_n\beta^r(\theta_n)e^{j\theta_n}|^2$, which can be formally described as

$$\begin{aligned}&\underset{\boldsymbol{\theta}}{\text{maximize}}\ f(\boldsymbol{\theta})\\&\text{subject to}\ \theta_n\in\Phi_K,\ n=1,2,\ldots,N\end{aligned}\tag{1}$$

where

$$f(\boldsymbol{\theta})=\left|\beta_0 e^{j\alpha_0}+\sum_{n=1}^{N}\beta_n\beta^r(\theta_n)e^{j(\alpha_n+\theta_n)}\right|^2,\tag{2}$$

$\beta_n\geq 0,\ n=0,1,\ldots,N$, $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_N)$, and $\alpha_n\in[-\pi,\pi)$ for $n=0,1,\ldots,N$.

The objective term $f(\boldsymbol{\theta})$ can alternatively be written as $f_1(\mathbf{w})=|h_0+\sum_{n=1}^{N}h_n\mathrm{w}_n|^2$ and is therefore connected to the generic K-ary discrete quadratic program (QP). In [1], the globally optimum solution for this problem was achieved in the least number of steps for a uniform discrete phase shift set with unit RIS gains, i.e., $\omega_k=2\pi/K, k=1,\ldots,K$, and $\beta^r(\theta_n)=1$, for $\theta_n\in[-\pi,\pi)$. In [2], the problem was addressed for a nonuniform discrete phase shift set with adjustable RIS gains, i.e., $\beta^r\in[0,1]$ being arbitrarily adjustable. In this paper, we provide an optimal algorithm for this problem, where there is a dependence between the phase shift selection of the RIS element and its gain.

[1]In this paper, we define $\oplus$ and $\ominus$ to choose from RIS phase shift indexes from 1 to K as follows. For $k_1,k_2\in\{1,\ldots,K\}$, $k_1\oplus k_2=k_1+k_2$ if $k_1+k_2\leq K$ and $k_1\oplus k_2=k_1+k_2-K$, otherwise. Similarly, for $k_1,k_2\in\{1,\ldots,K\}$, $k_1\ominus k_2=k_1-k_2$ if $k_1>k_2$ and $k_1\ominus k_2=K+k_1-k_2$, otherwise.

In the following sections, first, we define the PDA model for the discrete phase shift structures given an RIS phase range. Second, we provide necessary and sufficient conditions to achieve the globally optimum solution with discrete phase shifts based on arbirary $\mathbf{W}_K$. Then, we provide an optimum algorithm that employs these conditions.

II. Phase Dependent Amplitude and the RIS Phase Range

Practical RISs have certain limitations due to hardware constraints that do not favor the use of continuous phase shifts or the assumption of unit gains across all elements. Also, they allow only a certain range of phases to be used [3], [4]. First, we observe that it is difficult to realize continuous phase shifts, and control bits should be used to implement a finite set of phase shifts for practicality. Second, in practice, RISs can provide a certain range of phase shifts that potentially results in a nonuniform structure for the set of discrete phase shifts [3]. Finally, due to hardware limitations, the reflection amplitudes of the RIS are not necessarily constant over all the elements, and actually depend on the selected phase shift for each element [4]. Specifically, we focus on the following practical constraints:

- Using discrete phase shifts that are within the RIS phase range.
- Considering variable gains over RIS elements that depend on the selected phase shift, i.e., the PDA model.

We will first explain the discrete phase shift structure. Then, we will refer to the closed form equation from the literature for a practical RIS gain model.

A. The RIS Phase Range and the Discrete Phase Shifts

The RIS phase range $R \in [0, 2\pi]$ represents the phase-shifting capability of the RIS, allowing phase shift selections that are in the range $[-R/2, R/2]$, without loss of generality [2], [3]. We make the discrete phase shifts equally separated within the RIS phase range based on the proven optimality in [2]. This also helps present structured numerical results that are simple to follow.

When it comes to equally separated discrete phases over the RIS phase range, there are two possibilities: the discrete phases may or may not be placed uniformly over the unit circle.[2] The condition that determines whether the equally separated discrete phases will be uniform or not depends on the RIS phase range being sufficient or limited, i.e., $R \geq 2\pi\frac{K-1}{K}$ and $R < 2\pi\frac{K-1}{K}$, respectively. Therefore, the discrete phase shift set is determined as follows:

$$\Phi_K = \begin{cases} \{-\frac{R}{2}, \frac{R}{K-1} - \frac{R}{2}, \ldots, (K-2)\frac{R}{K-1} - \frac{R}{2}, \frac{R}{2}\} \\ \qquad\qquad\qquad\qquad \text{for } R < 2\pi\frac{K-1}{K}, \\ \{0, \omega', \ldots, (K-1)\omega'\} - \frac{(K-1)\omega'}{2} \\ \qquad\qquad\qquad\qquad \text{for } R \geq 2\pi\frac{K-1}{K}, \end{cases} \tag{3}$$

where $\omega' = \frac{2\pi}{K}$. Fig. 2 and Fig. 3 represent the second line and first line of (3), respectively. In Fig. 2 all arc lengths are the same and equal to $2\pi/K$. Whereas, in Fig. 3, the arc lengths within the range R are the same and are equal to $R/(K-1)$. In (3), subtracting $\frac{(K-1)\omega'}{2}$ ensures that the uniform discrete phase shifts are symmetric, just like the assumption with the nonuniform discrete phase shifts for compliance. In [2, Section VI] it was shown that the placement of ϕ_ks as in Fig. 2 and Fig. 3 corresponds to an optimal placement when $\beta^r(\phi_k) = 1$, $k = 1, \ldots, K$, for $R \geq 2\pi\frac{K-1}{K}$ and $R < 2\pi\frac{K-1}{K}$, respectively. Therefore, without loss of generality, we determine our discrete phase shifts in the range of $[-\pi, \pi)$ as $-\pi \leq \phi_1 < \cdots < \phi_K < \pi$. For the performance analysis in this paper, discrete phase shifts will be placed uniformly as shown in Fig. 2, if R is large enough, i.e., $R \geq 2\pi\frac{K-1}{K}$. Otherwise, we will use the approach in Fig. 3, i.e., $-\pi \leq \phi_1 < \cdots < \phi_K = \phi_1 + R < \pi$ when $R < 2\pi\frac{K-1}{K}$.

We remark that, while we provide the possible selection strategies for the discrete phases, these approaches are not required. The contribution of these assumptions is to enable us to quantify the RIS performance for settings that are suitable for practical scenarios. Without loss of generality, the optimum algorithm proposed in this paper will work with any arbitrary discrete phases to solve the general K-ary QP problem.

B. Phase Dependent Amplitude Constraint and A Practical Model

The PDA attenuation is caused by the RIS elements when they reflect signals, where the resulting gain of the RIS element depends on the selected phase shift of the element [4], [5]. In general, $\beta^r(\theta_n) : [-\pi, \pi] \to [0, 1]$ can be an arbitrary real-valued function that represents the phase-dependent gain of each RIS element. The analysis and algorithmic framework presented in this paper do not rely on a specific parametric form of $\beta^r(\theta_n)$. Rather, a structural condition on the resulting set of RIS coefficients $\mathbf{W}_K$ is required to achieve global optimality in the least number of steps, with the proposed linear-time algorithm. Specifically, for $K \geq 3$, we require that any three consecutive coefficients in $\mathbf{W}_K$ form a convex triplet, when the triplet spans an angle less than π. In other words, the middle point does not fall under the line connecting the outer points, thereby preserving convexity. Under this condition, our search algorithm achieves the globally optimal solution in linear time. When the condition is not met, the algorithm can still yield a benchmark with strong approximation behavior. Fig. 4 illustrates the PDA function $\beta^r(\theta_n) : [-\pi, \pi] \to [0, 1]$, where the $\beta^r(\theta_n)$ values are plotted for $\theta_n \in [-\pi, \pi]$, and

[2]Note that the terms uniform and nonuniform depend on the range over which they are defined. In this paper, we use the term nonuniform to mean the distribution over the full phase range $[-\pi, \pi)$ is nonuniform, or not equally separated.

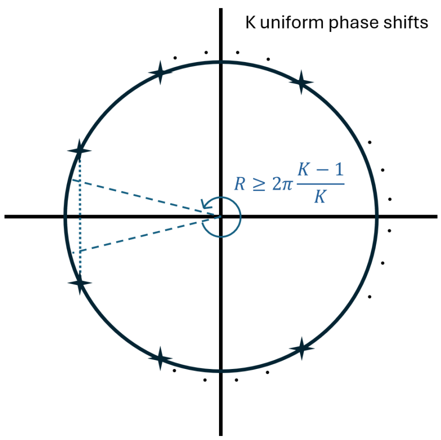

Fig. 2: Placement of uniformly distributed discrete phase placement for a sufficient phase range.

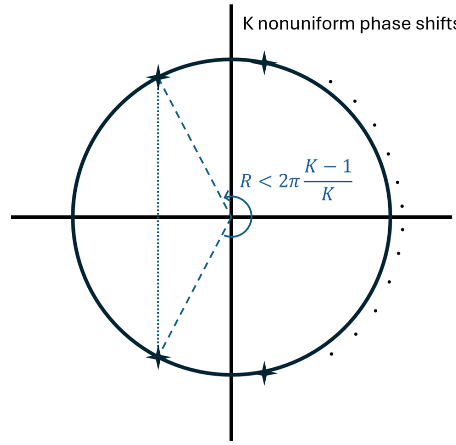

Fig. 3: Placement of nonuniformly distributed discrete phase placement for a limited phase range.

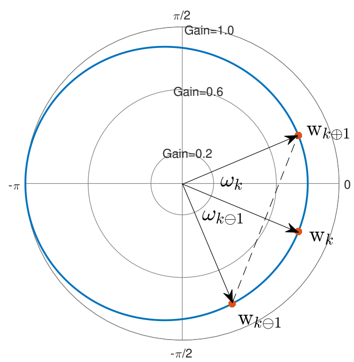

Fig. 4: An arbitrary PDA curve on the complex plane with three instances from $\mathbf{W}_K$ for $K \geq 3$.

the RIS coefficients. The PDA curve is sampled at certain locations to have the resulting RIS coefficients set $\mathbf{W}_K$. Fig. 4 also shows an example of local convexity among a triplet.

A particularly useful case arises when the continuous gain function $\beta^r(\theta_n)$ traces a strictly convex, smooth, closed curve in the complex plane. For such curves, any arc that spans less than π radians lies entirely on the outer side of the chord connecting its end points and exhibits a strictly outward-bulging shape. As a result, for any three consecutive RIS coefficients $\mathrm{w}_{k\ominus 1}$, w_k, and $\mathrm{w}_{k\oplus 1}$ sampled in order along the curve, the middle point w_k necessarily lies strictly outside the chord joining $\mathrm{w}_{k\ominus 1}$ and $\mathrm{w}_{k\oplus 1}$. This follows from the properties of strictly convex curves, which ensure that the arc between any two points lies entirely on one side of the corresponding chord [6], [7]. It is worth noting that strict convexity of $\beta^r(\theta_n)$ is not a necessary condition for the proposed method to apply. Even when the underlying curve is not strictly convex, the sampled discrete set $\mathbf{W}_K$ often satisfies the convexity condition, depending on the placement of the RIS coefficients. In particular, when the discrete phases are uniformly spaced and $K \leq 4$, the convexity of $\mathbf{W}_K$ is typically preserved. As a result, the proposed linear-time algorithm remains applicable and achieves the global optimum in most practical settings.

1) A Practical PDA Model for RIS: To analyze the discrete beamforming performance of an RIS in a practical context, we use the mathematical relation for an RIS model provided in [4] to generate performance results, where the same model [8], [9], or similar ones [10], [11], are used in published works on RIS. According to [4], the relation between the phase-dependent gain $\beta^r(\theta_n)$ and the phase shift θ_n for the n-th RIS element is

$$\beta^r(\theta_n) = (1 - \beta^r_{min}) \left(\frac{\sin(\theta_n - \phi^r) + 1}{2} \right)^{\alpha^r} + \beta^r_{min} \quad (4)$$

where β^r_{min} is the minimum amplitude corresponding to the maximum attenuation and therefore quantifies the level of attenuation, e.g., the lower the β^r_{min} the higher the attenuation. Variables β^r_{min}, ϕ^r, and α^r are practical hardware-related parameters that depend on the implementation of the RIS and are independent of the channel. In (4), $\phi^r \geq 0$ determines the phase shift for which the maximum attenuation occurs. It is defined as the difference between $-\pi/2$ and θ_n such that $\beta^r(\theta_n = \phi^r - \pi/2) = \beta^r_{min}$. The parameter $\alpha^r \geq 0$ controls the steepness of the PDA curve, governing the transition from the maximum gain at $\phi^r + \pi/2$ to the minimum gain at $\phi^r - \pi/2$. The selection of ϕ^r essentially rotates the whole PDA curve on the complex plane. Choosing $\pi/2$ makes the PDA curves symmetric around the real axis, similar to discrete phase shift selections. We use $\phi^r = \frac{\pi}{2}$ in the remainder of this paper. This assumption helps with the flow and, as will be discussed in the sequel, the calculation of the approximation ratio for the RIS. Among the PDA parameters, β^r_{min} plays the most significant role, therefore we focus our analysis on this parameter. With this, it is readily shown in [4] that the selection of α^r has a marginal impact on the overall performance compared to β^r_{min}. Therefore, we fix $\alpha^r = 1.6$ to get the results in this paper, similar to [4].

2) Discussion on Convexity of the Practical RIS Model: The shape of the PDA curve is primarily determined by two parameters: β^r_{min} and α^r, while ϕ^r induces a rotation of the entire curve. We closely examined the convexity of the PDA curve. Under the PDA constraint, a convex PDA profile is generally maintained, except in extreme cases with significant attenuation, e.g., when $\beta^r_{min} < 0.4$ and $\alpha^r < 1.2$. Similarly, the convexity is also preserved for $\beta^r_{min} < 0.4$, provided that α^r does not become excessively large, e.g., $\alpha^r \leq 2$.

To ensure convex RIS coefficients across a wide range of scenarios, we point to the values $\beta^r_{\min} \in [0.4, 1]$ and $\alpha^r \in [1.4, 2]$. This range is both practical and effective for preserving convexity, and the choice of α^r offers flexibility due to its marginal impact on system performance [4]. Finally, we remark that, prior works addressing the problem in (1) with uniform and nonuniform discrete phase shifts, i.e., [1] and [2], respectively, exhibited global convexity in the RIS coefficients due to the assumption of constant gain at the RIS elements, i.e., having a circular PDA curve.

We assume convex RIS coefficients in the remainder of this paper, based on the analysis in this section. Having the problem and the constraints established, we will next provide an optimal algorithm to solve the received power maximization problem in (1).

III. Optimal Solution with Discrete Phase Shifts

In this section, we provide the necessary and sufficient condition to achieve the global optimum solution. Based on the convexity discussions on the RIS coefficients $\mathbf{W}_K$ in Section II, we will employ the necessary and sufficient conditions to get the global optimum in linear time. Note that we want to maximize $|h_0 + \sum_{n=1}^{N} h_n \beta^r(\theta_n) e^{j\theta_n}|^2$ where $\theta_n \in \Phi_K$ and $h_n = \beta_n e^{j\alpha_n}$ for $n = 0, 1, \ldots, N$, $\beta_n \geq 0$, and $\alpha_n \in [-\pi, \pi)$. Let θ^*_n for $n = 1, \ldots, N$ be the discrete phase shift selections that give the global optimum. Define g as

$$g = h_0 + \sum_{n=1}^{N} h_n \beta^r(\theta^*_n) e^{j\theta^*_n}. \tag{5}$$

Let $\mu = g/|g|$ such that $|g| = g e^{-j\angle\mu}$. Similar to the condition in [2], we define the following lemma.

Lemma 1: For an optimal solution $(\theta^*_1, \theta^*_2, \ldots, \theta^*_n)$ for the received power maximization problem given in (1), it is necessary and sufficient that each θ^*_n satisfy

$$\theta^*_n = \arg\max_{\theta_n \in \Phi_K} \beta^r(\theta_n) \cos(\theta_n + \alpha_n - \underline{/\mu}) \tag{6}$$

for an arbitrary Φ_K.

Proof: We can rewrite $|g| = g e^{-j\angle\mu}$ as

$$\begin{aligned} |g| &= \beta_0 e^{j(\alpha_0 - \underline{/\mu})} + \sum_{n=1}^{N} \beta_n \beta^r(\theta^*_n) e^{j(\alpha_n + \theta^*_n - \underline{/\mu})} \qquad (7) \\ &= \beta_0 \cos(\alpha_0 - \underline{/\mu}) + j\beta_0 \sin(\alpha_0 - \underline{/\mu}) \\ &\quad + \sum_{n=1}^{N} \beta_n \beta^r(\theta^*_n) \cos(\theta^*_n + \alpha_n - \underline{/\mu}) \\ &\quad + j \sum_{n=1}^{N} \beta_n \beta^r(\theta^*_n) \sin(\theta^*_n + \alpha_n - \underline{/\mu}). \qquad (8) \end{aligned}$$

Because $|g|$ is real-valued, the second and fourth terms in (8) sum to zero, and

$$|g| = \beta_0 \cos(\alpha_0 - \underline{/\mu}) + \sum_{n=1}^{N} \beta_n \beta^r(\theta^*_n) \cos(\theta^*_n + \alpha_n - \underline{/\mu}), \tag{9}$$

from which (6) follows as a necessary and sufficient condition for the lemma to hold, since $\beta_n \geq 0$ for $n = 1, \ldots, N$. ∎

We remark that, unlike our previous works on the received power maximization problem [1], [2], maximizing $|g|$ in (9) is actually difficult with continuous phases. With discrete phases, one could at least attempt an exhaustive search to find the global optimum. Whereas, with continuous phases, finding the global optimum requires taking the partial derivatives of (9) with respect to each θ_n. However, $\underline{/\mu}$ is also a function of θ_n and this complicates the derivation significantly. Exhaustive search would be prohibitive as it would take $\mathcal{O}(K^N)$ steps where RISs are considered to have hundreds or even thousands of elements N. Yet, we will come up with a linear-time algorithm in the remainder of this section, enabling the received power maximization with discrete phase shifts.

Note that, in order to use the necessary and sufficient conditionsto achieve the global optimum with discrete phases, we need to know $\underline{/\mu}$, which comes from the optimal discrete phases that we do not know yet. Unless we have another condition, there are infinitely many possible $\underline{/\mu}$ values over the range $[-\pi, \pi)$. To make this into a finite set of possibilities, we start by observing the objective term in (6). The angle $(\theta_n + \alpha_n - \underline{/\mu})$ inside the cosine corresponds to the angle between the rotated channel vector and optimum $|g|$. Therefore, the overall value $\beta^r(\theta_n) \cos(\theta_n + \alpha_n - \underline{/\mu})$ corresponds to the projection of the rotated and scaled channel vector on the resulting sum. In other words, it is necessary and sufficient that for each RIS element, we need to maximize the length of the projection of the rotated and scaled channel vector. For better understanding, we use the simplified vector diagram in Fig. 5 to illustrate a decision boundary. For the n-th RIS element, we use $(\underline{/\mu} - \alpha_n)$ to represent the optimal direction and omit the channel gain β_n as it is independent from the selected phase shifts. The boundary between the two options will be the dashed red line that is drawn perpendicular from the origin to the line connecting the two RIS coefficients. If $(\underline{/\mu} - \alpha_n)$ is before this line, $\theta_n = \phi_{k\ominus1}$ is chosen. Similarly, if $(\underline{/\mu} - \alpha_n)$ is after this line, $\theta_n = \phi_k$ is chosen. To come up with a finite set of options for $\underline{/\mu}$, we need to find the boundary between each adjacent RIS coefficient, which amounts to K boundaries per element. Let s_k for $k = 1, \ldots, K$ denote the angle of these boundaries and let Δ_k be the angle from s_k to ϕ_k, then we have

$$\beta^r(\phi_k) \cos(\Delta_k) = \beta^r(\phi_{k\ominus1}) \cos(\omega_{k\ominus1} - \Delta_k),$$

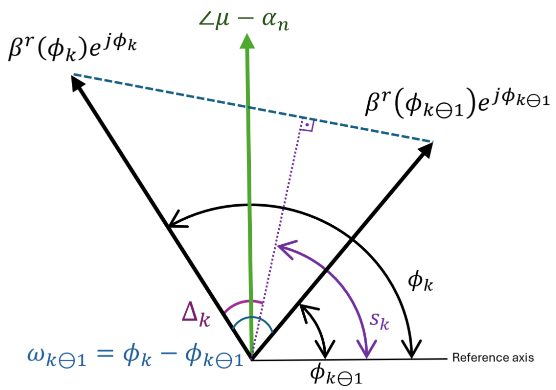

Fig. 5: A simple case of the new decision boundary. Decision boundary: Purple - at angle s_k - for $\phi_{k\ominus1} \rightarrow \phi_k$.

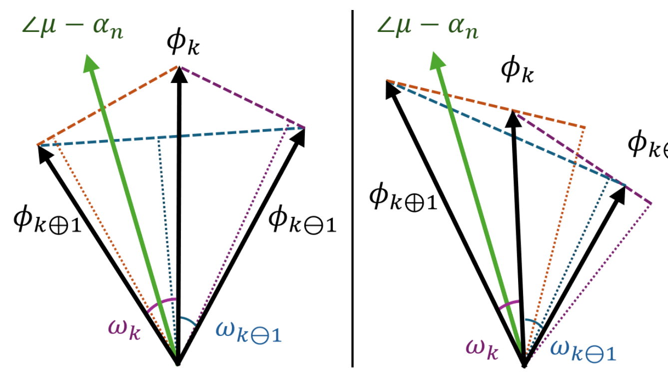

Fig. 6: An illustration for the convexity for the RIS coefficients. Decision boundaries: Purple - at angle s_k - for $\phi_{k\ominus1} \rightarrow \phi_k$; Orange - at angle $s_{k\oplus1}$ - for $\phi_k \rightarrow \phi_{k\oplus1}$; Blue - for $\phi_{k\ominus1} \rightarrow \phi_{k\oplus1}$.

meaning

$$\tan(\Delta_k) = \frac{\beta^r(\phi_k) - \beta^r(\phi_{k\ominus1})\cos(\omega_{k\ominus1})}{\beta^r(\phi_{k\ominus1})\sin(\omega_{k\ominus1})},$$

and

$$\Delta_k = \arctan\left(\frac{\beta^r(\phi_k) - \beta^r(\phi_{k\ominus1})\cos(\omega_{k\ominus1})}{\beta^r(\phi_{k\ominus1})\sin(\omega_{k\ominus1})}\right),$$

and, as a result,

$$s_k = \phi_k - \arctan\left(\frac{\beta^r(\phi_k) - \beta^r(\phi_{k\ominus1})\cos(\omega_{k\ominus1})}{\beta^r(\phi_{k\ominus1})\sin(\omega_{k\ominus1})}\right). \quad (10)$$

Therefore, given the boundaries at s_k for the n-th RIS element, we define the following sequence of complex numbers for each $n = 1, 2, \ldots, N$ to represent each and every boundary on the unit circle

$$s_{nk} = e^{j\left(\alpha_n + \phi_k - \arctan\left(\frac{\beta^r(\phi_k) - \beta^r(\phi_{k\ominus1})\cos(\omega_{k\ominus1})}{\beta^r(\phi_{k\ominus1})\sin(\omega_{k\ominus1})}\right)\right)} \quad (11)$$

for $k = 1, 2, \ldots, K$.

To show the simplified version of the boundaries when there is no gain attenuation, let $\beta^r_{min} = 1$ to have unit gain among all the RIS elements, that is, $\beta^r(\phi_{k\ominus1}) = \beta^r(\phi_k)$. From the half-angle identity, this would result in $\arctan\left(\frac{1-\cos(\omega_{k\ominus1})}{\sin(\omega_{k\ominus1})}\right) = \frac{\omega_{k\ominus1}}{2}$ and $s_{nk} = \exp(j(\alpha_n + \phi_k - \frac{\omega_{k\ominus1}}{2}))$, which turn out to be the same decision boundaries as in [2]. This shows that the decision boundaries and the respective solutions in [2] with the ideal gain assumption are a special case of the non-ideal gains considered in this paper.

We provide an additional sufficiency condition on θ_n^* using s_{nk} in (11), which will help us reduce the search space of μ, and come up with a linear time algorithm to achieve the global optimum solution, similar to [1], [2]. For this purpose, we assume for $K \geq 3$ that any three consecutive coefficients in $\mathbf{W}_K$ follow a convex curve as shown in Fig. 6, i.e., with respect to the origin, the coefficient in the middle is above the line connecting the adjacent coefficients. We note that this is actually a realistic assumption rather than an optimistic one, as discussed in Section II. Let us define $\mathrm{arc}(a : b)$, for any two points a and b on the unit circle C, to be the unit circular arc with a as the initial end and b as the terminal end in the counterclockwise direction, with the two endpoints a and b being excluded.

Proposition 1: A sufficient condition for $\theta_n^* = \phi_k$, independent of K, is

$$\mu \in \mathrm{arc}(s_{nk} : s_{n,k\oplus1}). \quad (12)$$

Proof: Assume μ satisfies (12). Then,

$$\angle\mu \in (\alpha_n + \phi_k - \Delta_k, \alpha_n + \phi_{k\oplus1} - \Delta_{k\oplus1}). \quad (13)$$

where $\Delta_k = \arctan(\frac{\beta^r(\phi_k) - \beta^r(\phi_{k\ominus1})\cos(\omega_{k\ominus1})}{\beta^r(\phi_{k\ominus1})\sin(\omega_{k\ominus1})})$. By taking the negation of (13) and adding θ_n and α_n, we get

$$\theta_n + \alpha_n - \angle\mu \in (\theta_n + \Delta_{k\oplus1} - \phi_{k\oplus1}, \theta_n + \Delta_k - \phi_k). \quad (14)$$

Now, consider each case when $\theta_n \in \{\ldots, \phi_{k\ominus1}, \phi_k, \phi_{k\oplus1}, \ldots\}$ and compute the projection of the rotated and scaled channel through the n-th RIS element onto the optimal direction $\angle\mu$. We define a contribution term to represent this projection as

$$C(\theta_n, \mu) \triangleq \beta^r(\theta_n)\cos(\theta_n - (\angle\mu - \alpha_n)). \quad (15)$$

Fig. 7 provides a visual illustration of the aforementioned projection, excluding the channel gain term β_n, as it scales all phase shift selections equally. Note that, $C(\theta_n, \mu)$ correspond to the objective term in *Lemma 1*. Consider the two edge cases where $(\angle\mu - \alpha_n)$ is close to either s_k or $s_{k\oplus1}$. In these scenarios, the definition of s_k in (10) guarantees that $C(\phi_{k\ominus1}, \mu) < C(\phi_k, \mu)$ and $C(\phi_{k\oplus1}, \mu) < C(\phi_k, \mu)$, respectively. Therefore, given $(\angle\mu - \alpha_n) \in (s_k, s_{k\oplus1})$, i.e., $\mu \in \mathrm{arc}(s_{nk} : s_{n,k\oplus1})$, any phase shift selection other than $\theta_n = \phi_k$ will evidently result in a lower projection value. Therefore, the contribution $C(\theta_n = \phi_k, \mu)$ is the maximum among all possible choices, ensuring that (12) satisfies the necessary and sufficient condition, thus completing the proof. ■

Finally, to operate with Proposition 1, we will eliminate duplicates among s_{nk} and sort to get $e^{j\lambda_l}$ such that $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_L < 2\pi$ as in [1] and [2]. To achieve the optimum

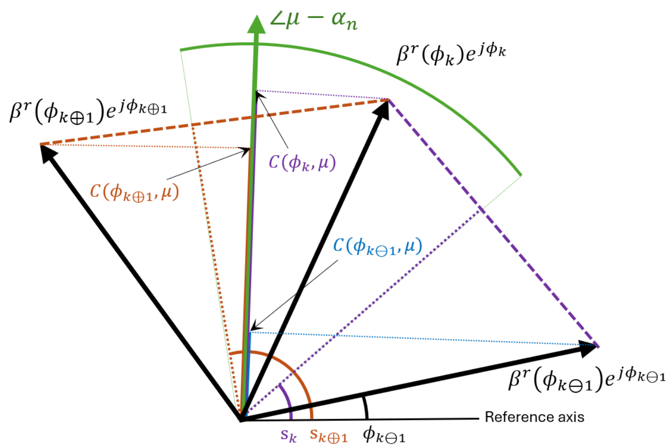

Fig. 7: An illustration for the optimality of $\theta_n^* = \phi_k$ given $\mu \in \text{arc}(s_{nk} : s_{n,k\oplus 1})$. Decision boundaries: Purple - at angle s_k - for $\phi_{k\ominus 1} \to \phi_k$; Orange - at angle $s_{k\oplus 1}$ - for $\phi_k \to \phi_{k\oplus 1}$.

solution in linear time, instead of assigning each θ_n with *Lemma 1* for each candidate μ, we will utilize *Proposition 1* to develop a search algorithm with elementwise updates. For this purpose, we need to track μ switching from one arc to another, i.e.,

$$\mu \in \text{arc}(e^{j\lambda_l} : e^{j\lambda_{l+1}}) \to \mu \in \text{arc}(e^{j\lambda_{l+1}} : e^{j\lambda_{l+2}}), \qquad (16)$$

and update the respective θ_n as given by the sufficiency condition in *Proposition 1*. With this, the elementwise update rule can be defined as

$$\mathcal{N}(\lambda_l) = \{\{n', k'\} | \angle s_{n'k'} = \lambda_l\}. \qquad (17)$$

The optimum algorithm starts by intializing a candidate $\angle\mu$, say $\angle\mu = 0$. At first, the discrete phases θ_n for $n = 1, \ldots, N$ are initialized with *Lemma 1*. Then μ traverses the unit circle in the counterclockwise direction. As μ moves along the unit circle and jumps over a boundary $e^{j\lambda_{l+1}}$, θ_n' will be updated for every $\{n', k'\} \in \mathcal{N}(\lambda_{l+1})$ according to the update rule in (17) as

$$\theta_{n'} \to \phi_{k'}, \quad \{n', k'\} \in \mathcal{N}(\lambda_{l+1}). \qquad (18)$$

Therefore, it is sufficient to consider $L \leq NK$ steps to find the global optimum, where only one or a few phase shifts are updated. This procedure is specified under Algorithm 1, which is a generalized version of Algorithm 1 from [2] to work with the PDA constraint, or in general for arbitrary RIS coefficients $\mathbf{W}_K$ that are locally convex. We present the cumulative distribution function (CDF) results for SNR Boost [12] in Fig. 8 for uniform discrete phases with $K = 4$ and $\beta_{min}^r \in \{0.2, 0.5, 0.8\}$. We compare Algorithm 1 with the exhaustive search results as a numerical validation of the optimality for $N = 10$. The CDF plots are generated with $1,000$ channel realizations where both Algorithm 1 and exhaustive search ran over the same realization in each step. It is clear that the results of Algorithm 1 perfectly align with the exhaustive search results, which are optimum. In

Algorithm 1 Generalized Algorithm 1 [2] for Phase-Dependent Amplitude

Initialization: Compute s_{nk} and $\mathcal{N}(\lambda_l)$ as in equations (11) and (17), respectively.
Set $\angle\mu = 0$. For $n = 1, 2, \ldots, N$, calculate and store
$$\theta_n = \arg\max_{\theta_n \in \Phi_K} \beta^r(\theta_n)\cos(\theta_n + \alpha_n - \angle\mu).$$
Set $g_0 = h_0 + \sum_{n=1}^N h_n \beta^r(\theta_n)e^{j\theta_n}$, $\texttt{absgmax} = |g_0|$.
for $l = 1, 2, \ldots, L-1$ **do**
  For each double $\{n', k'\} \in \mathcal{N}(\lambda_l)$, let $\theta_{n'} = \phi_{k'}$.
  Let
$$g_l = g_{l-1} + \sum_{\{n',k'\}\in\mathcal{N}(\lambda_l)} h_{n'}\left(\beta^r(\theta_n)e^{j\theta_n} - \beta^r(\phi_{k'\ominus 1})e^{j(\phi_{k'\ominus 1})}\right)$$
  if $|g_l| > \texttt{absgmax}$ **then**
    Let $\texttt{absgmax} = |g_l|$
    Store θ_n for $n = 1, 2, \ldots, N$
  end if
end for
Read out θ_n^* as the stored θ_n, $n = 1, 2, \ldots, N$.

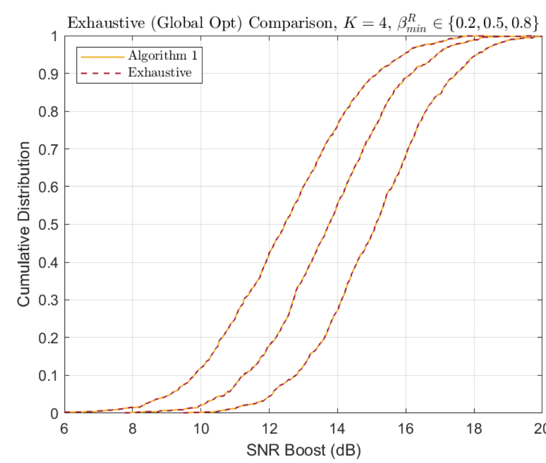

Fig. 8: CDF plots for SNR Boost with Algorithm 1 and exhaustive search, for $K = 4$ and $\beta_{min}^r \in \{0.2, 0.5, 0.8\}$, $N = 10$.

Fig. 9, we present the results obtained using Algorithm 1 with $K = 4$ discrete phase shifts for $\beta_{\min}^r \in 0.2, 0.5$ and $R \in \pi, 2\pi$. Two values of R are considered to represent uniform and nonuniform phase-shift cases, e.g., $K = 4$ with $R = \pi$ corresponds to a nonuniform discrete phase configuration. The CDFs are computed for $N = 16$, 64, and 256 elements over 10,000 channel realizations. The results show that the performance decreases with smaller $\beta_{\min}^r$ and R values. Furthermore, Fig. 9 indicates that the performance loss caused by amplitude attenuation becomes more significant for larger N, while the impact of R is larger for smaller array sizes.

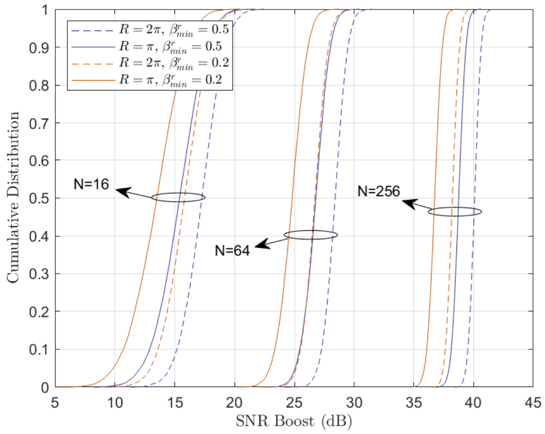

Fig. 9: CDF plots for SNR Boost with Algorithm 1, using $K = 4$ discrete phase shifts, for $\beta^r_{min} \in \{0.2, 0.5\}$ and $R \in \{\pi, 2\pi\}$.

IV. Convergence to Optimality and Complexity

In this section, we briefly analyze the complexity of Algorithm 1 and compare it with the optimal algorithms proposed in our earlier works. In Algorithm 1, the main for-loop (Steps 4–11) consists of $\sum_{l=1}^{L} \mathcal{O}(|\mathcal{N}(\lambda_l)|) = \mathcal{O}(NK)$ operations, where each step involves two vector additions to update a single element. Additionally, the initialization phase (Steps 2–3), which computes the initial coefficients and stores the corresponding gain using *Lemma 1*, requires N vector additions. As a result, the total computational cost of Algorithm 1 amounts to $N(2K + 1)$ vector additions.

Table 1 presents a comparison of Algorithm 1 with the optimal algorithms from [1] and [2], which handle the uniform and nonuniform discrete phase shift cases, respectively. When the gains of RIS coefficients are independent of the phase, i.e., constant gain, the complexity reduces to $N(K + 1)$ vector additions, while in the adjustable gain case it becomes $N(K+2)$, as reported in [2]. Furthermore, for strictly uniform discrete phase shifts, the complexity can be further reduced to N vector additions [1].

It is worth noting that the higher complexity of Algorithm 1 is expected, as the previous algorithms are special cases of the more general setting considered in this work. Without Algorithm 1, solving the K-ary quadratic programming (QP) problem under the PDA constraint would require exponential complexity.

V. Conclusion

In this work, we investigated the received power maximization problem for RIS-assisted communications under phase-dependent amplitude (PDA) constraints and discrete phase shifts that are within a limited RIS phase range. We established the necessary and sufficient conditions to achieve optimality in convex amplitude-phase configurations and proposed a linear-time search algorithm that guarantees convergence within NK steps. This work complements existing studies on received power maximization by addressing discrete beamforming under practical hardware constraints, such as phase-dependent amplitude attenuation. In that regard, it provides an extension to our prior works [1], [2]. Extensions of this work on algorithms with quantization and approximation ratio of discrete phase shifts can be found in [13].

Table 1: Comparison of Algorithm 1 with the optimum algorithms in earlier works.

	Search Steps	Time Complexity	RIS Coefficients
Opt. [1]	$\leq N$	$\mathcal{O}(N)$	Const. Gain Uniform
Opt.-1 [2]	$\leq NK$	$\mathcal{O}(N(K+1))$	Const. Gain Nonuniform
Opt.-2 [2]	$\leq N(K+1)$	$\mathcal{O}(N(K+2))$	On/Off Gain Nonuniform
Algo. 1	$\boldsymbol{\leq NK}$	$\boldsymbol{\mathcal{O}(N(2K+1))}$	PDA

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