Quantum Error Corrected Fidelity Routing Design for Long-distance Quantum Networks

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***Abstract*—The use of quantum teleportation in future quantum networks creates a completely new way for communication. As the quantum link length increased, the fidelity decreased, owing to the loss of quantum information between distant quantum nodes. This paper conducts an exploration of quantum fidelity routing design that is intended to specifically focus on the applicability of quantum error correction (QEC) as the scale of the network expands. Our research involves proposing a novel Quantum-maximum Fidelity miNimum Dijkstra's (Q-FIND) algorithm incorporated with QEC, which is designed to not only improve the long-distance link fidelity, but also to further reduce the network latency under the conditions of a noisy quantum channel. By using Qiskit, we examine the performance of proposed routing algorithm. Our simulation result reveals that the significance of QEC becomes more prominent in long-distance quantum networks.**

***Keywords*—Quantum Entanglement; Quantum Error Correction; Stabilizer Code; Fidelity Routing Design; Quantum Networks**

I. Introduction

As we move toward the emerging quantum networking age, a potential of quantum teleportation can be used for transferring unknown quantum states between distant nodes over an optical fiber link. Quantum entanglement serves as the cornerstone in the foundational proposal for quantum teleportation and the ensuing concept of quantum entanglement swapping, deepening our comprehension of this intricate quantum phenomenon [1]. A core aspect of quantum information theory, quantum entanglement, describes the condition where interacting particles can no longer be defined individually by their properties, but by the collective properties of the entire system. In a quantum network, various quantum nodes — consisting of quantum processors and memories — are interconnected via the optical fiber links [2]. These nodes are equipped to generate, store, exchange, and process quantum information in the form of *Qubits*. When two distant quantum nodes (source and destination) need to share information, the network sets up an entanglement connection between them, allowing the transmission of polarization quantum states of photons. However, the photons emitted by most quantum memories have visible or near-infrared wavelengths, and such photons are easily fragile and lost when they travel through optical fiber links. I. Marcikic et al. accomplished quantum teleportation of qubits within a 2-km standard telecommunication fiber, in which the transmission distance of a quantum state on the order of kilometers was first achieved in 2003 [3]. In 2012, the quantum teleportation over 100-km was implemented by the research groups of J. Yin, et al. [4] and X.S. Ma, et al. [5], seperately. Later, ground-to-satellite quantum teleportation with a single photon over 1,400 km was accomplished, in which the quantum communication can be realized at a global scale in 2017 [6]. Very recently, entanglement distribution has also been achieved over telecommunication fibers and analyzed retrospectively. Yet, to fully use entanglement over long-distance quantum network links it is mandatory to know it is available at the quantum nodes before the entangled state decays. T. van Leent, et al. [7] demonstrated heralded entanglement between two independently trapped single rubidium atoms generated over optical fiber links with a length up to 33 km.

On the other hand, noisy quantum channels introduce unwanted errors mainly through quantum entanglement between the information qubit and its environment. This leads to information leakage from the defined two-level qubit space into a larger Hilbert space [8]. Many quantum error correction (QEC) protocols have been proposed to address these errors [9]. In this paper, we propose and examine a combined quantum-error-corrected and quantum routing algorithm. A notable contribution of our work is the demonstration of feasible QEC if the noisy quantum channel is known, despite of the current studies of the quantum routing design in noiseless quantum channels. The following is explained the paper outline. Section II presents the background and motivation of this research. Section III introduces the system model in quantum networks. In Section IV, the proposed routing algorithm is further described. Section V elaborates the evaluation scenario, results and discussions of this research work. Last, we summarize our research works in Section VI.

II. Background and Motivation

Quantum applications often necessitate the establishment of an entanglement connection between quantum nodes. A single

S-D pair, which signifies a pair of quantum nodes with the intention to set up an entanglement connection. Creating end-to-end entanglement connections in quantum networks requires considering three unique operations: entanglement generation, purification, and entanglement swapping. Physical entanglement generation occurs between two controllable quantum nodes connected to an intermediate station, or a heralding station, over the optical fiber links. This can be achieved using different hardware platforms, such as nitrogen-vacancy centers in diamond [10]. Once generated, the entangled pair can be stored in the quantum nodes' memories and used to establish an entanglement connection or teleport an arbitrary single qubit state. Entanglement purification, implemented using controlled NOT (CNOT) gates or optically using polarizing beamsplitters [11], merges two low-fidelity Bell pairs into a higher-fidelity one. The post-purification fidelity can be calculated using the following formula [12]:

$$\overline{F} = \frac{F^2}{F^2 + (1-F)^2} \tag{1}$$

where F represents the fidelity of two Bell pairs during the purification operation, and $\overline{F}$ is the resultant fidelity. This purification process can be recursively applied to achieve, in principle, arbitrarily high fidelity. Entanglement swapping is an attractive approach to connect quantum nodes and establish an entanglement connection. This operation uses a quantum repeater to convert two one-hop entanglements into a direct entanglement between the nodes. By repeating the swapping operations, multi-hop entanglement connections along a path of repeaters carrying entangled pairs can be established. However, the imperfect measurements on the quantum repeater lead to the degradation of multi-hop entanglement fidelity during the swapping operation. As the fidelity of entangled pairs on different quantum channels varies, different routing paths result in distinct end-to-end entanglement connection fidelities after swapping, posing a challenge for designing high fidelity-guaranteed entanglement routing.

We realize that the performance study of quantum fidelity routing design in the quantum networks is not enough and needs more investigate. Thus, our motivation of this paper is to further investigate the quantum error corrected fidelity routing design for long-distance quantum networks. The objective of this paper is to propose novel Quantum-maximum Fidelity miNimum Dijkstra's algorithm incorporated with quantum error correction (Q-FIND/QEC) routing for improving the long-distance link fidelity while reducing the network latency under the conditions of noisy quantum channel.

III. System Model

A. Noisy Quantum Channel Model

Many studies assume that the fidelity routing design is in an ideal scenario, in which the quantum channel is no noise [13]. However, in practice, we must take into account the more realistic scenario where the noise channels are present. In this paper, we assume the qubit states are prepared at a sender's end and then transmitted through noisy quantum channels. We construct a parametric noisy quantum channel that exhibits similar characteristics to an *amplitude damping channel* (ADC) model [14] by using the open-source Quantum Information Science Kit (Qiskit) simulator [15], to introduce arbitrary noise into the qubits, with the noise level parameterized by a single variable. The qubits transmitted through the noisy quantum channel can be considered an open system, interacting with the environment during transmission [9]. The ADC model's energy relaxation from an excited state to the ground state. The evolution of this state, in conjunction with the environment under the ADC model can be expressed as: $U|\phi_s\rangle|0_e\rangle = \alpha|00_e\rangle + \beta|\sqrt{\gamma}\rangle|01_e\rangle + \beta|\sqrt{1-\gamma}\rangle|01_e\rangle$. The U is a unitary matrix that can be written as:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \sqrt{\gamma} & 0 \\ 0 & -\sqrt{\gamma} & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where $\gamma \in [0,1]$ is a flexible parameter. Next, we make a Z-basis measure on the auxiliary qubit and only keep the resulting state if the measurement outcome is $|0\rangle$, and at this point, the resulting state is: $|\psi\rangle = \frac{1}{N}(\alpha|0\rangle + \beta\sqrt{1-\gamma}$, where $N = \sqrt{\alpha^2 + \beta^2(1-\gamma)}$ is a normalization factor. This process has a success probability of $p_s = N^2$, in which it delivers the parameterized noisy quantum channel. To better simulate the real-world conditions, we then discard our explicit knowledge of γ. Instead, we work under the assumption that we only have statistical knowledge of the quantum channel model. More specifically, we assume γ lies between 0 to 1 and follows a Gaussian distribution, with an expected mean value $\gamma = 0.5$ for the experiments. Note that we will never use the explicit value of γ in any of the experiments conducted. While this process approximates some real-world channels, we do not claim it perfectly models any specific real-world channel.

In the quantum networks, a quantum teleportation through a quantum link can be solved numerically in the Lindblad form [16]. The fidelity as a function of decoherence time and angles of unknown θ and ϕ of the quantum state to be teleported can be written as:

$$F = \mu(1 + e^{-\sqrt{d(A,B)}/L}) \tag{3}$$

where μ denotes the value of average fidelity decays is 0.5. L signifies the attenuation length with a distance $d(A,B)$ between two nodes A and B.

B. Quantum Error Correction Model

Fidelity estimation is the initial critical step, where we evaluate the noisy quantum channel — the medium for data transmission. The objective of this step is to ascertain the original fidelity of every link within the network. The fidelity F of a quantum state ρ, with respect to a pure state $|\psi\rangle$, can be calculated using the following equation [12]:

$$F = \langle\psi|\rho|\psi\rangle \tag{4}$$

For a noisy quantum channel, we consider an entangled link $E(A,B)$ with a distance of $d(A,B)$. Assuming a pernode error probability p_e (which includes the effective logical error probability and other residual errors in the nodes), the entanglement fidelity can be reformulated as [17]:

$$F = (1 - p_e)^{2(d(A,B)+1)-2} \tag{5}$$

Fidelity estimation not only offers a preliminary assessment but also prepares the ground for the subsequent vital step in QEC. This process is designed to optimize the fidelity of the network by leveraging the data from fidelity estimation to pinpoint and rectify errors, thereby enhancing the overall fidelity. The core idea of QEC is to encode a quantum state $|\psi\rangle$ into a larger Hilbert space using an encoding operation E_i (Pauli X, Y, Z operation). In the event of an error E, it can be detected and corrected using the error correction operation R_j. The objective is to meet the Knill-Laflamme conditions: $R_j E_i E |\psi\rangle = C_{ij} |\psi\rangle$, where C_{ij} are some constants, and the sum over j of C_{ij}^2 is independent of i for all $|\psi\rangle$ [18]. Once the QEC is performed, the data reflecting the improved fidelity is stored. This data serves as a valuable resource for future reference and use, contributing to the optimization of the network's performance. It can be used to fine-tune the error correction process, making it more efficient and effective in maintaining high fidelity.

C. Quantum Circuit Representation

The quantum circuit under examination, as depicted in Fig. 1, incorporates the noisy quantum channel and utilizes a 5-qubit Stabilizer code for error correction. In this configuration, qubit q_0 serves as the logical qubit, holding the information to be protected by the stabilizer. Meanwhile, qubits q_1, q_2, q_3 and q_4 are designated as physical qubits, employed to encode and safeguard the logical information, and s denotes the syndrome qubits. A unitary gate (UG) is derived from the unitary matrix U and is used to parameterize the noisy quantum channel. Also, the error correction process in this circuit is bifurcated into encoding and error correcting phases. In the encoding phase, quantum gates, such as CNOT and Hadamard (H) gates, are applied to the physical qubits. This action encodes the logical qubit across the five qubits, thereby securing the information. During the error correction part, correction operations are performed based on identified error syndromes. These operations employ Pauli operators (X, Y, Z) to detect and correct errors. By applying these operators to the affected qubits, the logical qubit is restored to its correct state, ensuring the integrity of the encoded information.

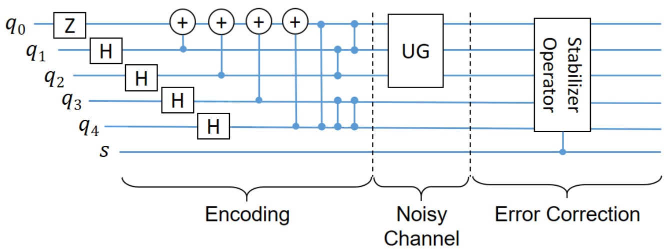


Fig. 1: QEC circuit under noisy quantum channel is constructed by Qiskit

Algorithm 1 Q-FIND/QEC Routing Algorithm

Input: $G = (V, E, C)$, F_i^{th}, request R_i and $\langle s_i, d_i \rangle$;
Output: $P_{i,j}(s_i, d_i)$, $D_{i,j}^{pur}$, $F_j(s_i, d_i)$, $T_{i,j}^{EXT}$

Step 1 Initialization:
Calculate *Purification Cost Table* for $(u, v) \in E$;
Delete all edges (u, v) from G, if $F_j(s_i, d_i) < F^{th}$;
Construct auxiliary graph $G^a = (V, E^a, C^a, Cost)$
Step 2 Fidelity Guaranteed Operation:
for $i = 1 : u, j = 1 : v$ **do**
  $F(i, j) \leftarrow$ applying the QEC to improve fidelity;
  if $F(i, j) > F'(i, j)$ **then**
    Update $F(i, j)$;
  else
    $i + 1, j + 1$;
  end if
end for
for $j = 1 : R_j$ **do**
  Step 3 Minimum Hop Path Selection:
  $P_{i,j}(s_i, d_i)$,$U(P_{i,j}(s_i, d_i)) \leftarrow$ using Dijkstra's algorithm to determine the shortest path;
  if no available path for $\langle s_i, d_i \rangle$ **then**
    break;
  end if
  Step 4 Maximum End-to-end Fidelity Path Selection:
  for $(u, v) \in P_{i,j}(s_i, d_i)$ **do**
    if $F(u, v) > F'(u, v)$ **then**
      Update $F(u, v)$;
    end if
  end for
  $Q \leftarrow P_{i,j}(s_i, d_i)$, $cost(P_{i,j}(s_i, d_i))$, $D_{i,j}^{pur}$;
  Step 5 Throughput Update:
  while $Q.pop! = null$ **do** $\leftarrow$ throughput updating
    Find $W_{i,j}^{min}$ along the path $P_{i,j}(s_i, d_i)$ in G^a
    if $W_{i,j} > 1$ **then**
      Subtract $min\{W_{i,j}^{min}, R_i\} \times (N_{i,j}^{pur}(u, v) + 1)$ on each $(u, v) \in P_{i,j}(s_i, d_i)$ from C^a in G^a;
    end if
    $T_{i,j}^{EXT} \leftarrow$ Calculate expected throughput of each edge $(u, v) \in P_{i,j}(s_i, d_i)$;
    Output $P_{i,j}(s_i, d_i)$, $D_{i,j}^{pur}$, $min\{W_{i,j}^{min}, R_i\}$
    $F_{i,j}(s_i, d_i)$ and delete this solution from Q;
    if $\sum_j T_{i,j}^{EXT}(s_i, d_i) \geq R_i$ **then**
      terminate;
    end if
  end while
end for

IV. Proposed Q-FIND/QEC Routing

In this section, we concentrate on a single S-D pair routing problem. We introduce Q-FIND, i.e., Quantum-maximum Fidelity miNimum Dijkstra's algorithm. The primary aim of Q-FIND is to discern the shortest routing path and make optimal purification using minimum hop links, based on the principles of Q-LEAP [13]. A significant advantage of Q-FIND is its ability to decrease network latency. While Q-LEAP has shown potential in tracing a single routing path of ‘optimum quality’ to achieve the necessary fidelity threshold for a single S-D pair, it has a disadvantage in terms of high hop count or long latency. To address this, we have engineered a low-latency routing algorithm incorporated with QEC, named Q-FIND/QEC, as shown in Algorithm 1. The basic idea behind Q-FIND/QEC, which takes more comprehensive approach to the latency reduction and fidelity improvement. In particular, Q-FIND/QEC examines all feasible shortest path options (measured in terms of the number of hops), selecting the path with highest fidelity. This simultaneous focuses on minimizing path length and maximizing fidelity is distinguished to Q-LEAP.

V. Evaluation Studies

A. Simulation Scenario and Condition

In this section, we conducted a series of extensive simulations to comprehensively evaluate the proposed routing algorithms that incorporate QEC techniques under the long-distance quantum networks. These simulations were ran using Python 3.10 on a desktop computer with a hardware specification of Intel Core i7 4790 3.6 GHz CPU and 32 GB DDR3 RAM, operating on a 64-bit version of Windows 10 environment. For this evaluation, the network topology of the Japan Phonics Network Model (JPNM) [19], was configured with 48 nodes, 82 links, and a quantum link capacity of 50 qubits/slot. The simulation parameters and settings are listed in Table I. In our simulation, we conducted fidelity estimation and improvement with QEC before routing algorithm. In addition to performance evaluation, we also monitored latency and throughput metrics, to assess the real-time applicability of our algorithms. We examined the resilience of our algorithms under quantum noise conditions, assessing how well they could maintain high fidelity while minimizing latency.

B. Results and Discussions

Fig. 2(a) and Fig. 2(b) depict the comparative performance of the proposed Q-FIND and modified Q-LEAP algorithms, both before and after the implementation of QEC. It is observed that throughput decreases rapidly as the fidelity threshold increases, and the latency correspondingly surges. This can largely be attributed to the characteristics of the JPNM, which is both large-scale and long-distance. The expansive distances involved in the JPNM significantly and negatively impact end-to-end fidelity. It is noteworthy that when the number of hops is set to 2, the average end-to-end link distance reaches an impressive 299.6 km within the JPNM. This vast distance inevitably introduces more noise and error into the quantum channel, thereby necessitating more purification operations to achieve the desired fidelity level, and in turn, escalating the overall latency. We observe that the Q-FIND/QEC algorithm again manifests superior performance, achieving the most optimal outcomes at a fidelity threshold of 0.6. At this juncture, it displays a noteworthy enhancement in throughput by 59.2% and a decrease in latency by 11.3% as compared to the Q-FIND algorithm.

TABLE I: Simulation Parameters and Settings

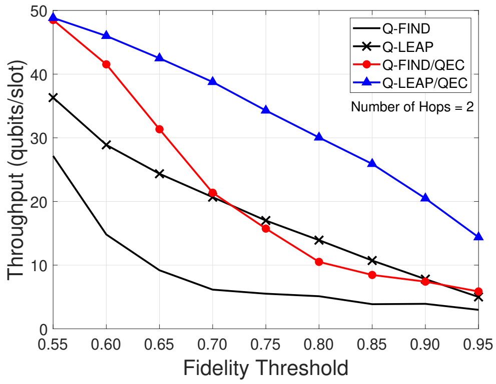
Parameter	Value
Number of quantum nodes	48
Number of quantum links	82
Distance between nodes	10 to 673 km
Number of requests	50
Quantum link capacity	50 qubits/slot
Quantum error correction	5-qubit Stabilizer code
Quantum channel model	ADC model ($\gamma = 0.5$)
Number of trails	10,000 times

Fig. 2(c) and Fig. 2(d) present a detailed evaluation of the performance of the proposed Q-FIND and Q-LEAP algorithms at a fidelity threshold of 0.90, both before and after the implementation of QEC. It should be noted that the term “No. of Hops” refers to the minimum number of hops between the source and destination nodes. It is a clear downtrend can be observed for both Q-LEAP and Q-FIND algorithms in terms of throughput. Meanwhile, latency levels peak, taking considerably longer times. This phenomenon can be attributed to the complexity and scale of the network, coupled with an increased fidelity threshold that demands more purification operations. Moreover, it is important to underline the key role the number of hops plays, as higher numbers introduce more possibilities for noise and errors, directly impacting the overall fidelity and latency of the network.

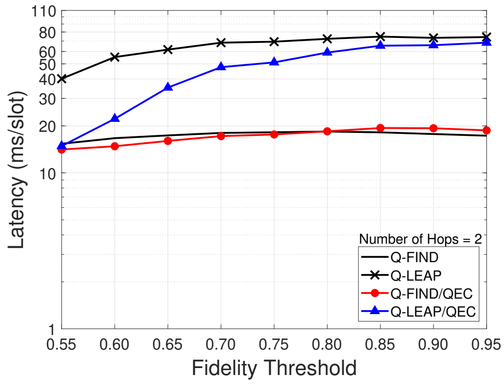
Both the algorithms, whether applied the QEC or not, demand additional processing time to meet this end-to-end fidelity requirement. Based on the Fig. 2, the conclusion can be drawn that QEC leads to noticeable improvements in performance, particularly for moderate fidelity requirements. In summary, the significance of QEC becomes more prominent in large-scale, long-distance network models like JPNM. In other words, in the JPNM, where quantum states traverse vast distances and more noise-introducing links, QEC serves as a pivotal component to maintain and improve the fidelity of the network, thereby making it indispensable. Although purification operations can effectively manage and improve the fidelity in the JPNM, it is unavoidable to increase the latency.

VI. Conclusion

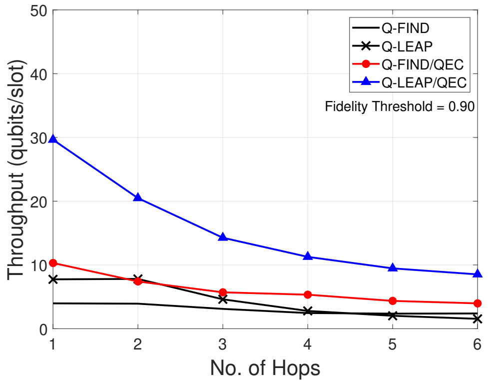
In this paper, we presented a detailed evaluation of the performance of our proposed Q-FIND/QEC algorithm, and the previously established Q-LEAP algorithm. This evaluation was conducted under two different scenarios: one without and another one with the implementation of QEC. From our analysis, it was clear that the application of QEC significantly improved the overall performance of both algorithms. Also, the Q-FIND algorithm demonstrated a distinguished ability to further reduce latency, whether QEC was implemented or not compare with Q-LEAP. However, results also highlighted that when a high end-to-end fidelity is required, the improvements in latency and throughput performance become less. For future works, we will conduct an experiment with more effective QEC codes to further refine the fidelity and overall performance of our quantum network algorithm.



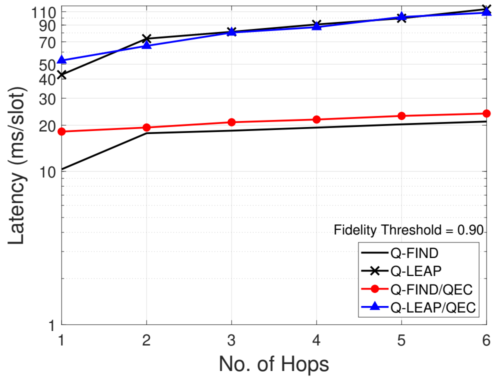
(a) Throughput vs. Fidelity Threshold



(b) Latency vs. Fidelity Threshold



(c) Throughput vs. No. of Hops



(d) Latency vs. No. of Hops

Fig. 2: Performance comparison for single S-D pair in terms of throughput and latency

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