Intermittent Markov Frequency-Hopping Entropy Rate

Hyuck M. Kwon

***Abstract*—The entropy rate of a random process shows how the entropy of the sequence grows with time. The entropy rate can be used to estimate the complexity of a random process. In this paper, the entropy rate of multiple, intermittent frequency hopping (FH), stationary, and Markov model (MM) random process sources, is presented in a closed form. Also, an upperbound of the entropy rate of the multiple hidden Markov model (HMM) sequences corresponding to the multiple MM sources is presented.**

***Index Terms*—entropy rate, frequency hopping, intermittent, Markov, hidden Markov.**

I. INTRODUCTION

In the current paper, a multiple-access network system model used in [1] is considered where multiple, intermittent FH, stationary, and Markov random process sources are accessing. However, the research in the current paper is different from the one in [1], which studied sparse recovery of intermittent frequency hopping signals aided by direction of arrival angle. In the current paper, the entropy rate defined in [2], is derived mathematically for the multiple, intermittent FH, and stationary Markov random process sources used in [1]. The entropy rate is the time density of the average information in a random process. The entropy rate can be used to estimate the complexity of a random process [3].

II. SYSTEM MODEL

The frequency hopping is a repeated switching of frequencies during radio transmission according to a specified algorithm (e.g., a uniform FH in this paper), to minimize unauthorized interception or jamming of telecommunications. In addition, a radio transmission system including many radio transmitters using FH carriers intermittently transmits, e.g., indicative very short messages of status of sensors associated with the transmitters [1]. In operation, a time interval generator included in a transmitter generates pulses activating the transmitter at time intervals according to a predetermined algorithm (e.g., a first-order Markov model (MM) in this paper). When activated, the transmitter transmits a message at one (in this paper) or several different frequencies. The frequencies are changed according to a predetermined algorithm and preferably differ for each subsequent transmission [1].

Figure 1 shows a random walk diagram for a single-source intermittent FH sequence activity [1]. The source hops from its current frequency to another frequency uniformly every hop time interval, and then the source transmits its signals intermittently by following the MM process.

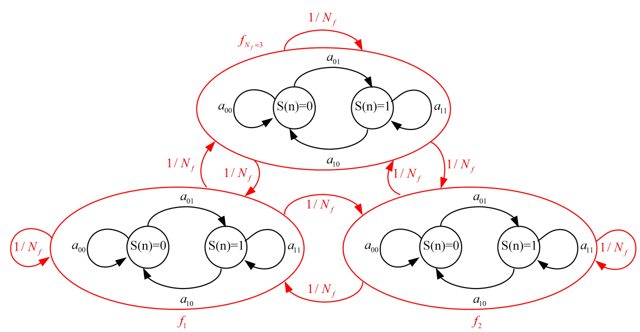

Fig. 1: Intermittent FH random walk with uniform FH probability $1/N_f$, e.g., $N_f = 3$ and stationary Markov transition matrix $A[a_{ij}]$, $i, j = 0, 1$ [1].

Figure 2 shows an example of intermittent FH sequences $X(n)$ versus hop time interval for source S_1 in red color, and for source S_2 in black color. In Figs. 1 and 2, N_f and N_s denote the number of FH frequencies and the number of sources, respectively. Also, $\{f_1, f_2, ...f_{N_f}\}$ and $\{S_1, S_2, ..., S_{N_s}\}$ denote, respectively, the corresponding sets of FH frequencies and the sources. In the current paper, the FH frequency of source l hops to frequency f_i with uniform probability $1/N_f$, $i = 1, ..., N_f$, $l = 1, ..., N_s$ at a hop time, and then source l determines its intermittent transmission activity at sample time n using a stationary MM [2]. This is referred to as an intermittent FH frequency. Multiple samples or one sample per hop can be considered during a hop interval. Let $S(n)$ represent the common stationary Markov switching process at sample time n for all source S_l, $l = 1, ..., N_s$:

$$S(n) \triangleq \begin{cases} 1 & \text{if Switch ON } at\ n \\ 0 & \text{if Switch OFF } at\ n \end{cases} \quad (1)$$

where the hopped frequency f_i is used and signals are transmitted when the switch is ON, and the not-hopped frequency

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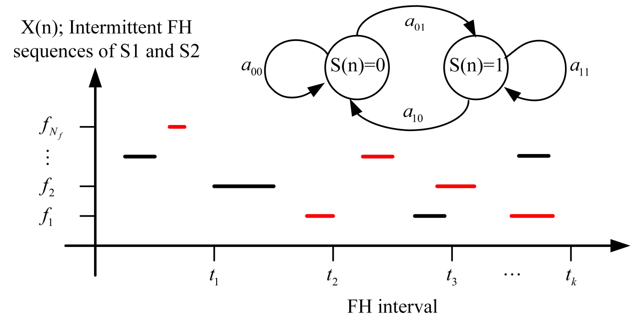

Fig. 2: Intermittent FH sequences $\mathbf{X}(n)$ of sources S_1 and S_2 with independent stationary Markov transition matrix $A[a_{ij}]$, $i, j = 0, 1$ [1].

f_c is used and signals are not transmitted when the switch is OFF.

Let a_{ij} denote the transition probability $Pr[S(n+1) = j|S(n) = i]$ from the current state $S(n) = i$ to the next state $S(n+1) = j$ in the stationary MM process, $i, j = 0, 1$ for all sources l, $l = 1, ..., N_s$.

Then, the MM stationary state probabilities can be written as [2] [Equation (4.8) on Page 73]

$$Pr(S(n)) = \begin{Bmatrix} \frac{a_{01}}{a_{01}+a_{10}} & \text{if} \quad S(n) = 1 \\ \frac{a_{10}}{a_{01}+a_{10}} & \text{if} \quad S(n) = 0 \end{Bmatrix} \tag{2}$$

Let $\mathbf{X}(n) \triangleq (X_1(1), ..., X_1(n), ..., X_{N_s}(1), ..., X_{N_s}(n))$ denote the multiple-source, joint, intermittent FH frequency signals at sampling time n when all sources transmit their intermittent FH signals independently by following the common Markov model process in (1).

The Shannon entropy in [2] [Equation (2.3) on Page 14] is employed, i.e., the entropy of a random variable X is given by $H(X) \triangleq E_X[\log(\frac{1}{p_X(X)})]$, where $p_X(X)$ is the probability mass (or density) function, and $E_X[X]$ is the expectation of a random variable X with respect to $p_X(X)$. In the next section, the Shannon entropy rate of the multiple-source joint intermittent FH random process $\mathbf{X}(n)$ will be derived.

III. ENTROPY RATE OF JOINT INTERMITTENT FH SEQUENCE

About the entropy rate of multiple-source, joint, intermittent FH sequence, the following theorem is stated:

***Theorem* 1.** The joint entropy rate of multiple-source and intermittent FH sequences with each source of a Markov model activity $A[a_{ij}]$ $i, j = 0, 1$, can be written as

$$\begin{aligned} H_J(\mathcal{X}) &\triangleq \lim_{n\to\infty} \frac{1}{n} H(\mathbf{X}(n)) \\ &= N_s \times \frac{a_{01}}{a_{01}+a_{10}} \left[\frac{a_{01}}{a_{01}+a_{10}} \frac{N_f - 1}{N_f} + a_{11} \frac{1}{N_f} \right] \\ &\quad \times \log \left[\frac{1}{\frac{1}{N_f} \left[\frac{a_{01}}{a_{01}+a_{10}} \frac{N_f-1}{N_f} + a_{11} \frac{1}{N_f} \right]} \right] \end{aligned} \tag{3}$$

where N_s and N_f are, respectively, the number of sources and the number of FH frequencies, and a_{ij} is the MM state transition probability: $a_{ij} = Pr(S(n+1) = j|S(n) = i)$ with $S(n)$ in (1).

Proof. The proof is completed through subsections *A* and *B*. In subsection *A*, the entropy rate of a single-source intermittent FH source is derived. In subsection *B*, the entropy rate of multiple-source, joint, and intermittent FH sequences is derived. ∎

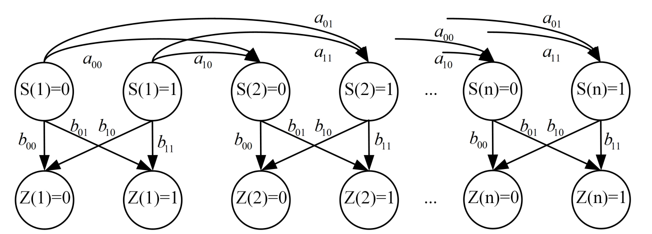

Fig. 3: An HMM $Z(n)$ with transition probability $b_{jk} \triangleq Pr(Z(n) = k|S(n) = j)$ where $S(n)$ is an MM of $A[a_{i,j}]$ [1].

An operator at each sample time n in Fig. 3 observes whether a hopped frequency f_i is active or inactive. Let $S(n)$ (not $X(n)$) and $Z(n)$ represent, respectively, the input and output random process of this observation operator with transition probability $b_{jk} = Pr(Z(n) = k|S(n) = j)$, $j, k = 0, 1$. The $S(n)$ is the true intermittent FH activity, and $Z(n)$ is an observed intermittent FH activity. The $Z(n)$ is a hidden Markov model (HMM) process. In subsection *C*, after the proof of Theorem 1, an upper bound of the entropy rate for the HMM process $Z(n)$ will be derived.

A. Single Source Intermittent FH Entropy Rate

***Theorem* 2.** The entropy rate $H(\mathcal{X})$ of a single-source intermittent FH frequency sequence $X(n)$ is equal to

$$\begin{aligned} H(\mathcal{X}) &\triangleq \lim_{n\to\infty} \frac{1}{n} H(X(1), ..., X(n)) = H(X(2)|X(1)) \\ &= \frac{a_{01}}{a_{01}+a_{10}} \left[\frac{a_{01}}{a_{01}+a_{10}} \frac{N_f - 1}{N_f} + a_{11} \frac{1}{N_f} \right] \\ &\quad \times \log \left[\frac{1}{\frac{1}{N_f} \left[\frac{a_{01}}{a_{01}+a_{10}} \frac{N_f-1}{N_f} + a_{11} \frac{1}{N_f} \right]} \right]. \end{aligned} \tag{4}$$

Proof. The entropy rate $H(\mathcal{X})$ is equal to the conditional entropy rate $H'(\mathcal{X}) \triangleq \lim_{n\to\infty} H(X(n)|X(n-1),...,X(1))$ from Theorem 4.2.1 in [2] because $X(n)$ is a stationary and MM process. Furthermore, $H'(\mathcal{X}) = \lim_{n\to\infty} H(X(n)|X(n-1))$ because $X(n)$ is a first-order Markov process. In the current paper, only the first-order MM is considered. Moreover, the entropy rate $H(\mathcal{X})$ is equal to the conditional entropy $H(X(2)|X(1))$ because $X(n)$ is stationary. Hence, $H(X(2)|X(1))$ can be written as

$$H(X(2)|X(1)) = \sum_{i=1}^{N_f} H(X(2)|X(1)=f_i)Pr(X(1)=f_i) \quad (5)$$

Let μ_i denote the probability that $X(n) = f_i$ at sample time n. Then, $\mu_i = Pr(X(1) = f_i)$ because $X(n)$ is stationary. The conditional entropy in (5) can be rewritten as

$$\begin{aligned} H(X(2)|X(1)) = &-\sum_{i=1}^{N_f} \mu_i \sum_{j=1}^{N_f} Pr(X(2) = f_j|X(1) = f_i) \\ &\times \log(Pr(X(2) = f_j|X(1) = f_i)). \end{aligned} \quad (6)$$

From the total probability,

$$\begin{aligned} \mu_i &= Pr(X(n) = f_i|S(n) = 1)Pr(S(n) = 1) \\ &\quad + Pr(X(n) = f_i|S(n) = 0)Pr(S(n) = 0) \\ &= Pr(X(n) = f_i|S(n) = 1)Pr(S(n) = 1) \\ &= Pr(X_h(n) = f_i)Pr(S(n) = 1) = \frac{1}{N_f}\frac{a_{01}}{a_{01}+a_{10}}, \end{aligned} \quad (7)$$

where $Pr(X_h(n) = f_i)$ is the probability of FH activity hopped to f_i at sample time n, and $Pr(X(n) = f_i)$ is the probability of both FH activity hopped to f_i and the MM switching activity to ON. The third equality in (7) is from $Pr(X(n) = f_i|S(n) = 0)Pr(S(n) = 0) = 0$, i.e., the event $\{X(n) = f_i\}$ given the MM OFF state $\{S(n) = 0\}$ cannot happen because $X(n)$ should be the non-hopped carrier frequency f_c instead of an FH frequency f_i if the MM switch is OFF. The fourth equality in (7) is because of the following: (a) the three events $\{X(n) = f_i, S(n) = 1\}$, $\{X(n) = f_i\}$, and $\{\{X_h(n) = f_i\} \cap \{S(n) = 1\}\}$ are equivalent, and (b) events $\{X_h(n) = f_i\}$ and $\{S(n) = 1\}$ are independent, i.e., the FH activity and MM switching activity are independent. Here, $\{X(n) = f_i\}$ is the event $\{X_h(n) = f_i\} \cap \{S(n) = 1\}$. Hence, $\{X(n) = f_i\} \neq \{X_h(n) = f_i\}$. Then, $\mu_{h,i} \triangleq Pr\{X_h(n) = f_i\} = \frac{1}{N_f}$ because of the uniform FH activity. This will be proven in (13). Thus, the conditional probability $Pr(X(2) = f_j|X(1) = f_i)$ in (6) can be rewritten as

$$\begin{aligned} &Pr(X(2) = f_j|X(1) = f_i) \\ &= Pr(X(2) = f_j|X(1) = f_i, f_j \neq f_i)Pr(f_j \neq f_i) \\ &\quad + Pr(X(2) = f_j|X(1) = f_i, f_j = f_i)Pr(f_j = f_i) \\ &= Pr(X(2) = f_j)Pr(f_j \neq f_i) \\ &\quad + Pr(X(2) = f_j|X(1) = f_i, f_j = f_i)Pr(f_j = f_i) \\ &= Pr((X_h(2) = f_j) \cap (S(2) = 1))Pr(f_j \neq f_i) \\ &\quad + Pr((X_h(2) = f_j) \cap (S(2) = 1)|(X_h(1) = f_i) \\ &\qquad \cap (S(1) = 1), f_j = f_i)Pr(f_j = f_i) \\ &= Pr(X_h(2) = f_j)Pr(S(2) = 1)Pr(f_j \neq f_i) \\ &\quad + Pr((X_h(2) = f_j) \cap (S(2) = 1)|(X_h(1) = f_i) \\ &\qquad \cap (S(1) = 1), f_j = f_i)Pr(f_j = f_i) \\ &= Pr(X_h(2) = f_j)Pr(S(2) = 1)Pr(f_j \neq f_i) \\ &\quad + Pr(X_h(2)=f_j)Pr(S(2)=1)|S(1)=1)Pr(f_j=f_i) \\ &= \frac{1}{N_f}\left[\frac{a_{01}}{a_{01}+a_{10}}\frac{N_f-1}{N_f} + a_{11}\frac{1}{N_f}\right] \end{aligned} \quad (8)$$

where the first equality is from the total probability, the second equality from conditional independence between $\{X(1) = f_i\}$ and $\{X(2) = f_j\}$ given $\{f_j \neq f_i\}$, the third equality from the equivalence between $\{X(1) = f_i\}$ and $\{\{X_h(1) = f_i\} \cap \{S(1) = 1\}\}$ and the equivalence between $\{X(2) = f_j\}$ and $\{\{X_h(2) = f_j\} \cap \{S(2) = 1\}\}$, and the fourth equality from the independence between the FH activity $\{X_h(2) = f_j\}$ and MM switching activity $\{S(2) = 1\}$. The fifth equality is from the following: The event $\{f_j \neq f_i\}$ happens with probability $\frac{N_f-1}{N_f}$. In this case, the conditional probability $Pr(S(2) = 1|S(1) = 1)$ becomes $Pr(S(2) = 1) = \frac{a_{01}}{a_{01}+a_{10}}$. If the FH frequency f_j at $n = 2$ and the FH frequency f_i at $n = 1$ are the same, i.e., if $\{f_j = f_i\}$, then the MM switching activities $S(1)$ and $S(2)$ are dependent due to the Markov chain. This event $\{f_j = f_i\}$ happens with probability $\frac{1}{N_f}$. In this case, the conditional probability $Pr(S(2) = 1|S(1) = 1)$ becomes a_{11}. These and the independence between $\{X_h(2) = f_j\}$ and $\{X_h(1) = f_i\}$ are used in the fifth equality in (8). The conditional entropy in (6) can be rewritten using (2) and (8) as

$$\begin{aligned} H(X(2)|X(1)) =& \frac{a_{01}}{a_{01}+a_{10}}\left[\frac{a_{01}}{a_{01}+a_{10}}\frac{N_f-1}{N_f} + a_{11}\frac{1}{N_f}\right] \\ &\times \log\left[\frac{1}{\frac{1}{N_f}\left[\frac{a_{01}}{a_{01}+a_{10}}\frac{N_f-1}{N_f} + a_{11}\frac{1}{N_f}\right]}\right]. \end{aligned} \quad (9)$$

Note that the conditional entropy in (9) can be approximated for a sufficiently large N_f as

$$H(X(2)|X(1)) \approx \left[\frac{a_{01}}{a_{01}+a_{10}}\right]^2 \log\left[\frac{N_f(a_{01}+a_{10})}{a_{01}}\right] \quad (10)$$

which is increasing in a logarithmic function as N_f increases.

From the connection loop in red, Fig. 1 for FH process, the stationary FH probability $\mu_{h,i}$ is equal to $1/N_f$ for all $i = 1, ..., N_f$. This is because

$$[\mu_{h,1} \cdots \mu_{h,N_f}] \begin{bmatrix} P_{11} & \dots & P_{1N_f} \\ \vdots & & \vdots \\ P_{N_f 1} & \dots & P_{N_f N_f} \end{bmatrix} = [\mu_{h,1} \cdots \mu_{h,N_f}] \quad (11)$$

by [2][Page 73], where

$$P_{ij} \triangleq Pr[X_h(n+1) = f_j | X_h(n) = f_i] = \frac{1}{N_f}, \quad (12)$$

for $i, j = 1 \dots N_f$. A solution for $\mu_{h,i}$ in (11) is

$$\mu_{h,1} = \dots = \mu_{h,N_f} = \frac{1}{N_f}. \quad (13)$$

Therefore, the proof is completed from (13). ■

B. Multiple-Source Intermittent FH Entropy Rate

When multiple intermittent FH sources are active, the joint entropy rate can be stated as follows:

***Theorem* 3.** The joint entropy rate of independent and identically distributed (i.i.d.) multiple-source intermittent FH sequences with each MM source, is equal to N_s times the individual entropy rate $H(\mathcal{X})$ of a single-source intermittent FH frequency sequence $X(n)$, i.e.,

$$H_J(\mathcal{X}) = N_s H(\mathcal{X}) \quad (14)$$

where N_s is the number of sources.

Proof.

$$\begin{aligned} H_J(\mathcal{X}) &\triangleq \lim_{n\to\infty} \frac{H(X_1(1), ..., X_1(n), ..., X_{N_s}(1), ..., X_{N_s}(n))}{n} \\ &= \lim_{n\to\infty} \frac{H(X_1(1), ..., X_1(n))}{n} \\ &\quad + \sum_{l=2}^{N_s} \frac{H(X_l(1), ..., X_l(n) | X_{l-1}(1), ..., X_{l-1}(n)}{n} \\ &= \sum_{l=1}^{N_s} \lim_{n\to\infty} \frac{1}{n} H(X_l(1), ..., X_l(n)) = N_s H(\mathcal{X}) \end{aligned} \quad (15)$$

where the entropy chain rule is used in the first equality, and then i.i.d. source conditions are used in the second and fourth equalities, respectively. ■

Remark 1: Both the entropy rate in (4) for the single-source intermittent FH sequence $X(n)$ and the joint entropy rate in (14) for i.i.d. multiple-source intermittent FH sequences $\mathbf{X}(n)$ are functions of MM switching parameters $a_{i,j}$, $i, j = 0, 1$ and the number of FH frequencies N_f and the number of sources N_s.

C. Hidden Markov Intermittent FH Entropy Rate for a Single Source

In this subsection, single-source intermittent FH activity is considered. The FH frequency is not estimated, but the FH activity is observed. The activity of the source is modeled as a binary state sequence $S(n)$. If the source is active, i.e., one of FH frequencies is active, then $S(n) = 1$, and otherwise, $S(n) = 0$. Then, $Z(n)$, which denotes ON-OFF FH activity, becomes an HMM process, where $Z(n) = 1$ and $Z(n) = 0$ represent that the observed intermittent FH activity is ON and OFF, respectively. Figure 3 shows the HMM model. The conditional observation probability given $S(n)$ is denoted as $b_{jk} \triangleq Pr(Z(n) = k | S(n) = j)$, $j, k = 0, 1$. The entropy rate of a general HMM is not typically available in a closed form. An upper bound of the entropy rate of the specific HMM model $Z(n)$ in Fig. 3 will be derived:

***Theorem* 4.** The entropy rate $H(\mathcal{Z})$ of an observed intermittent FH activity $Z(n)$ for a single MM source is less than or equal to $H(Z(1))$, i.e.,

$$\begin{aligned} H(\mathcal{Z}) &\triangleq \lim_{n\to\infty} \frac{1}{n} H(Z(1), ..., Z(n)) \le H(Z(1)) \\ &= \mathcal{H}\left(b_{01} \frac{a_{10}}{a_{01} + a_{10}} + b_{11} \frac{a_{01}}{a_{01} + a_{10}} \right) \end{aligned} \quad (16)$$

where $\mathcal{H}(\alpha)$ is the entropy function, i.e.,

$$\mathcal{H}(\alpha) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \left(\frac{1}{1 - \alpha} \right), \quad (17)$$

$$a_{ij} = Pr(S(n+1) = j | S(n) = i) \quad (18)$$

with $S(n)$ in (1) and

$$b_{jk} \triangleq Pr(Z(n) = k | S(n) = j), \quad (19)$$

for $i, j, k = 0, 1$.

Proof.

$$\begin{aligned} Pr(Z(1) = 1) &= Pr(Z(1) = 1 | S(1) = 0) Pr(S(1) = 0) \\ &\quad + Pr(Z(1) = 1 | S(1) = 1) Pr(S(1) = 1) \\ &= b_{01} Pr(S(1) = 0) + b_{11} Pr(S(1) = 1). \end{aligned} \quad (20)$$

From (2) and (20),

$$\begin{aligned} H(Z(1)) &= \mathcal{H}(Pr(Z(1) = 1)) \\ &= \mathcal{H}\left(b_{01} \frac{a_{10}}{a_{01} + a_{10}} + b_{11} \frac{a_{01}}{a_{01} + a_{10}} \right). \end{aligned} \quad (21)$$

Then, the joint entropy can be written as

$$\begin{aligned} H(Z(1), \dots, Z(n)) &= H(Z(1)) + \sum_{i=2}^{n} H(Z(i) | Z(i-1), ..., Z(1)) \\ &\le \sum_{i=1}^{n} H(Z(i)) = n H(Z(1)). \end{aligned} \quad (22)$$

The chain rule is used in the first equality in (22). Then, the fact that the conditional entropy is less than or equal to the unconditional entropy is used in the second equality. The last equality is from the identical probability distribution of $Z(i)$ and $Z(j)$. Therefore, the proof of (16) is completed using (21) and (22). ■

Note that the Hidden Markov process $Z(i)$ is a dependent random process in general. For example, $Z(i) = S(i)$ when $b_{jk} = 0$ for $j \neq k$ and the input Markov process $S(i)$ is a dependent process. This is why the upper bound was used in (22).

D. Joint Hidden Markov Intermittent FH Entropy Rate for Multiple Sources

In this section, multiple-source intermittent FH activities are considered. The activity of the lth source is modeled as a binary state sequence $S_l(n)$, $l = 1, ..., N_s$. If the lth source is active, then $S_l(n) = 1$, and otherwise, $S_l(n) = 0$. And the observed ON-OFF FH activity is denoted by $Z_l(n)$. Then, $Z_l(n)$ becomes an HMM process where $Z_l(n) = 1$ and $Z_l(n) = 0$ represent the observed intermittent FH activity ON and OFF, respectively. Figure 3 shows the HMM model. The conditional observation probability given $S_l(n)$ is denoted by $b_{l,jk} \triangleq Pr(Z_l(n) = k|S_l(n) = j)$, $j, k = 0, 1$ for $l = 1, ..., N_s$. An upper bound of the joint entropy rate of the HMM $\mathbf{Z}(n) \triangleq (Z_1(1), ..., Z_1(n), ..., Z_{N_s}(1), ..., Z_{N_s}(n))$ will be derived:

***Theorem* 5.** The joint entropy rate $H_J(\mathcal{Z})$ of multiple observed intermittent FH activities $\mathbf{Z}(n)$ for multiple MM sources is less than or equal to $N_s H(Z(1))$, i.e.,

$$\begin{aligned} H_J(\mathcal{Z}) &\triangleq \lim_{n\to\infty} \frac{1}{n} H(\mathbf{Z}(n)) \leq N_s \; H(Z_1) \\ &= N_s \; \mathcal{H}\left(b_{01}\frac{a_{10}}{a_{01}+a_{10}} + b_{11}\frac{a_{01}}{a_{01}+a_{10}}\right), \end{aligned} \tag{23}$$

where $\mathcal{H}(\alpha)$ is the entropy function, $a_{ij} = Pr(S(n+1) = j|S(n) = i)$ with $S(n)$ in (1), and $b_{jk} \triangleq Pr(Z(n) = k|S(n) = j)$, $i, j, k = 0, 1$.

Proof.

$$\begin{aligned} H_J(\mathcal{Z}) &= \lim_{n\to\infty} \frac{H(Z_1(1), ..., Z_1(n))}{n} \\ &\qquad + \sum_{l=2}^{N_s} \frac{H(Z_l(1), ..., Z_l(n)|Z_{l-1}(1), ..., Z_{l-1}(n)}{n} \\ &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{N_s} H(Z_l(1), ..., Z_l(n)) \\ &= \sum_{l=1}^{N_s} \lim_{n\to\infty} \frac{1}{n} H(Z_l(1), ..., Z_l(n)) = N_s H(\mathcal{Z}) \end{aligned} \tag{24}$$

where the entropy chain rule is used in the first equality, and then the property that the conditional entropy is less than or equal to the unconditional entropy is used in the second inequality, and the last equality is the entropy rate definition. ■

Remark 2: Both the upper bounds of the single-source HMM entropy rate $H(\mathcal{Z})$ in (16) and the joint multiple-source HMM entropy rate $H_J(\mathcal{Z})$ in (23) are functions of MM parameters $a_{i,j}$, HMM parameters $b_{i,j}$, $i, j = 0, 1$, and the number of sources N_s. However, both upper bounds are independent of the number of hopping frequencies N_f because the HMM model considered in Fig. 3 depends only on the MM switching activity $S(n)$ (not $X(n)$) and observation activity $Z(n)$.

IV. CONCLUSIONS

In this paper, the entropy rates, $H(\mathcal{X})$ and $H_J(\mathcal{X})$, for single- and multiple joint-source, intermittent FH sequences with each source of an MM activity, were derived in closed forms. It was found that the entropy rates are linearly increasing as the number of sources N_s increases. In addition, the entropy rates are increasing in a logarithmic function as the number of FH frequencies N_f increases. Furthermore, it was found that both entropy rates, $H(\mathcal{X})$ and $H_J(\mathcal{X})$, are functions of the MM switching activity parameters $a_{i,j}$. Then, upper-bounds of single- and joint multiple-source HMM entropy rates $H(\mathcal{Z})$, $H_J(\mathcal{Z})$, for the observed intermittent activity sequences $Z(n)$ and $\mathbf{Z}(n)$, were derived, and found to be less than or equal to, respectively, $H(Z(1))$ and $N_s H(Z(1))$. Moreover, it was found that both the upper bounds of $H(\mathcal{Z})$ and $H_J(\mathcal{Z})$ are functions of MM switching activity parameters, $a_{i,j}$, the HMM observation parameters $b_{i,j}$, and the number of sources N_s, but not functions of the number of hopping frequencies N_f.

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